

COHOMOLOGY THEORY FOR GENERAL SPACES

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1. Introduction

Many definitions have been given for extending the cohomology theory of complexes to arbitrary topological spaces. At present the two theories most commonly used are the Čech and singular theories [6]¹. A different definition was given in 1935 by J. W. Alexander [1] who constructed cohomology groups for a compact metric space using functions of sets of points in the space. Alexander stated that, with suitable choice of coefficient groups, the cohomology groups obtained by his construction were character groups of the Vietoris homology groups. Since the Čech groups for a compact metric space agree with the Vietoris groups, Alexander's construction gives the Čech cohomology groups for such a space. Inasmuch as the definition involving functions is simpler than the definition in terms of coverings usually used to obtain the Čech groups, a generalization of Alexander's definition applicable to arbitrary spaces is desirable. A. D. Wallace has suggested such a definition which is valid for any space and is even somewhat simpler than Alexander's definition.

This paper is concerned with the question of how the groups defined by Wallace compare with the cohomology groups obtained by the various constructions used at present. It is shown that on finite complexes the new groups coincide with those defined by means of the cellular structure. On compact Hausdorff spaces, the new groups agree with the Čech cohomology groups. This implies that they differ from the singular cohomology groups as it is known that there exist compact Hausdorff spaces on which the Čech and singular theories differ. On non-compact spaces, the groups do not agree with the Čech groups. For example, on an infinite locally finite complex they coincide with the cohomology groups based on infinite cochains, whereas C. H. Dowker [2] has shown that the Čech groups do not have this property. Since the singular groups of infinite locally finite complexes agree with those based on infinite cochains [3], the new groups coincide with the singular ones for such spaces. Thus the Wallace definition gives rise to an extension of cohomology from polyhedra to general spaces which is distinct from the known extensions.

The Wallace definition is presented in Section I together with the definitions of the relative cohomology groups $H^p(X, A)$ of X modulo a subset A , the homomorphisms $f^*: H^p(Y, B) \rightarrow H^p(X, A)$ induced by a continuous map $f: (X, A) \rightarrow (Y, B)$, and the coboundary homomorphism $\delta: H^{p-1}(A) \rightarrow H^p(X, A)$.

In Section II, the system (H^p, f^*, δ) is shown to be a cohomology theory in

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¹ Numbers in square brackets refer to the bibliography at the end of the paper.



that it satisfies the axioms of Eilenberg-Steenrod [4]. The definitions of Section I are thus justified by the verification of axioms in Section II. It follows from the axioms that the new groups coincide on finite complexes with the cohomology theory based on the cellular structure.

In Section III, it is proved that (H^p, f^*, δ) is continuous under passage to inverse limits for compact spaces. Cohomology theories which have this property are unique on the class of compact Hausdorff spaces. Since the Čech theory also satisfies this continuity condition [7], the two theories agree on compact spaces.

On locally finite complexes, it is shown that (H^p, f^*, δ) coincides with the cohomology theory based on infinite cochains. Thus the cohomology theory studied herein differs from both the Čech theory and the singular theory. However, it agrees with the Čech theory on compact spaces and with the singular theory on locally finite complexes.

The construction of these cohomology groups does not appear to suggest a way of constructing the corresponding homology groups. In Appendix A, a second (more complicated) construction is given. This construction is used to define the dual homology theory. A more general form of this construction was given in 1936 by Alexander [8].

In Appendix B, cup products of cocycles are discussed briefly.

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I. CONSTRUCTION OF THE COHOMOLOGY THEORY

2. Notation

Let X be a topological space. X^{p+1} will denote the $(p+1)$ -fold product of X with itself; that is, X^{p+1} is the set of points $(x_0, x_1, \dots, x_p)$ where $x_i \in X$, this set being topologized in the usual way. By way of abbreviation, x^{p+1} will be used to denote an arbitrary point of X^{p+1} . ΔX^{p+1} will denote the diagonal of X^{p+1} . In particular, $\Delta X^1 = X^1 = X$.

For $p \geq 1$ and $0 \leq i \leq p$, we define a projection

$$\pi_i: X^{p+1} \rightarrow X^p$$

$$\text{by} \quad \pi_i(x_0, x_1, \dots, x_p) = (x_0, \dots, \hat{x}_i, \dots, x_p).^2$$

It is then easy to verify that π_i has the following properties:

$$(2.1) \quad \pi_i \text{ maps } \Delta X^{p+1} \text{ into } \Delta X^p.$$

$$(2.2) \quad \pi_i = \pi_j \text{ on } \Delta X^{p+1}.$$

$$(2.3) \quad \pi_i \pi_j = \pi_{j-1} \pi_i \text{ if } i < j.$$

² $(x_0, \dots, \hat{x}_i, \dots, x_p)$ denotes the p -tuple consisting of the $(p+1)$ -tuple $(x_0, \dots, x_p)$ with x_i omitted.

If $A \subset X$, A^{p+1} is the set of points $(x_0, \dots, x_p) \in X^{p+1}$ such that $x_i \in A$ for all i .

In the remainder of this paper *function* will be used to denote an arbitrary mapping from one set to another, while *mapping* will be reserved for a continuous mapping from one topological space to another.

3. Definition of p -functions

Let G denote a fixed discrete abelian group. G will be used as coefficient group throughout the rest of this paper except for the appendices.

The group $\Phi^p(X)$ of p -functions, is defined to be the set of all functions ϕ from X^{p+1} to G with functional addition as the group operation. A p -function ϕ will be said to be *locally zero on X* if ϕ is zero on some neighborhood $N(\phi)$ of ΔX^{p+1} . If ϕ, ϕ' are both locally zero on X , then $\phi - \phi'$ is zero on $N(\phi) \cap N(\phi')$ showing that the set of p -functions which are locally zero on X forms a subgroup $\Phi_0^p(X) \subset \Phi^p(X)$.

4. Cohomology groups

If $\phi \in \Phi^p(X)$, its *coboundary* $\bar{\delta}\phi \in \Phi^{p+1}(X)$ is defined by

$$(\bar{\delta}\phi)(x^{p+2}) = \sum_{i=0}^{p+1} (-1)^i \phi(\pi_i(x^{p+2}))$$

and is a homomorphism

$$\bar{\delta}: \Phi^p(X) \rightarrow \Phi^{p+1}(X).$$

Its important properties are:

$$(4.1) \quad \bar{\delta}\bar{\delta} = 0.$$

$$(4.2) \quad \bar{\delta} \text{ maps } \Phi_0^p(X) \text{ into } \Phi_0^{p+1}(X).$$

To prove (4.1), let $\phi \in \Phi^p(X)$. Then

$$\begin{aligned} \bar{\delta}\bar{\delta}\phi(x^{p+3}) &= \sum_{i=0}^{p+2} (-1)^i \bar{\delta}\phi(\pi_i(x^{p+3})) = \sum_{i=0}^{p+2} (-1)^i \sum_{j=0}^{p+1} (-1)^j \phi(\pi_j \pi_i(x^{p+3})) \\ &= \sum_{\substack{1 \leq i \leq p+2 \\ j < i}} (-1)^{i+j} \phi(\pi_j \pi_i(x^{p+3})) + \sum_{\substack{0 \leq i \leq p+1 \\ j \geq i}} (-1)^{i+j} \phi(\pi_j \pi_i(x^{p+3})). \end{aligned}$$

Apply (2.3) to the first number of the right hand side and then substitute h, k for $i-1, j$. In the second member of the right hand side, substitute h, k for j, i . Then the two members differ in sign only, so the right hand side vanishes.

To prove (4.2), let $\phi \in \Phi_0^p(X)$ be zero on $N(\phi)$. Then $\bar{\delta}\phi$ is zero on $\bigcap_{i=0}^{p+1} \pi_i^{-1}(N(\phi))$ which is a neighborhood of ΔX^{p+1} by (2.1) and the fact that π_i is continuous.

A p -function ϕ will be said to be a p -cocycle if $\bar{\delta}\phi$ is locally zero. The set of p -cocycles will be denoted by $\Phi_z^p(X)$. Since $\Phi_z^p(X) = \bar{\delta}^{-1}(\Phi_0^{p+1}(X))$, $\Phi_z^p(X)$ is a subgroup of $\Phi^p(X)$.

If $p > 0$, a p -function ϕ will be said to be a p -coboundary if there is $\phi' \in \Phi^{p-1}(X)$ such that $\phi - \bar{\delta}\phi' \in \Phi_0^p(X)$. The set of p -coboundaries will be denoted by $\Phi_b^p(X)$.

Since $\Phi_B^p(X)$ is the span of $\delta(\Phi^{p-1}(X))$ and $\Phi_0^p(X)$, $\Phi_B^p(X)$ forms a subgroup of $\Phi^p(X)$. $\Phi_B^0(X)$ is defined to be the group consisting of the function which is zero everywhere; that is, $\Phi_B^0(X) = (0)^3$.

Using (4.1) and (4.2), it is trivial to verify that

$$\Phi_{00}^p(X) \subset \Phi_B^p(X) \subset \Phi_s^p(X) \subset \Phi^p(X) \quad \text{for all } p.$$

The p^{th} cohomology group $H^p(X)$ will be defined to be the factor group

$$H^p(X) = \Phi_s^p(X) / \Phi_B^p(X),$$

and β will denote the natural homomorphism

$$\beta: \Phi_s^p(X) \rightarrow H^p(X).$$

5. Induced homomorphisms of the functions

Let f be a function (not necessarily continuous) from X to Y . f will also be used to denote the function from X^{p+1} to Y^{p+1} defined by

$$f(x_0, \dots, x_p) = (f(x_0), \dots, f(x_p)).$$

The following properties of f are immediate:

$$(5.1) \quad f \text{ maps } \Delta X^{p+1} \text{ into } \Delta Y^{p+1}.$$

$$(5.2) \quad \pi_i f = f \pi_i.$$

If $\phi \in \Phi^p(Y)$, define $f^* \phi \in \Phi^p(X)$ by

$$(f^* \phi)(x^{p+1}) = \phi(f(x^{p+1})).$$

Then f^* is a homomorphism

$$f^*: \Phi^p(Y) \rightarrow \Phi^p(X)$$

with the following properties:

$$(5.3) \quad \text{If } f: X \rightarrow X \text{ is the identity, then } f^* \text{ is the identity.}$$

$$(5.4) \quad \text{If } f: X \rightarrow Y \text{ and } g: Y \rightarrow Z, \text{ then } (gf)^* = f^* g^*.$$

$$(5.5) \quad f^* \bar{\delta} = \bar{\delta} f^*.$$

(5.3) and (5.4) follow from the definition of f^* , and (5.5) follows from (5.2) and the definition of $\bar{\delta}$.

$$(5.6) \quad \text{If } f \text{ is continuous, then } f^* \text{ maps } \Phi_0^p(Y) \text{ into } \Phi_0^p(X).$$

³ To avoid the necessity of always considering the case $p = 0$ as an exception, the following definitions are added. Define $\Phi^p(X)$ for all $p < 0$ to be the group consisting solely of the zero element. For $p < 0$, define $\bar{\delta}: \Phi^p(X) \rightarrow \Phi^{p+1}(X)$ to be the trivial homomorphism. Then $\Phi_B^0(X)$ is the span of $\bar{\delta}(\Phi^{-1}(X))$ and $\Phi_0(X)$, so that $\Phi_B^p(X)$ is defined in the same way for all $p \geq 0$. These definitions will be used throughout the rest of the paper.

If $f: X \rightarrow Y$ is continuous, then $f: X^{p+1} \rightarrow Y^{p+1}$ is also continuous. Hence, if $\phi \in \Phi_0^p(Y)$ is zero on N , then $f^*\phi$ is zero on $f^{-1}(N)$ showing that $f^*\phi \in \Phi_0^p(X)$.

(5.7) If A is a subset of X and $i^*: A \rightarrow X$ is the identity map ($i(x) = x$ for $x \in A$), then i^* maps $\Phi^p(X)$ onto $\Phi^p(A)$.

If $\phi \in \Phi^p(A)$, define $\bar{\phi} \in \Phi^p(X)$ by

$$\bar{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in A^{p+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $i^*\bar{\phi} = \phi$ showing that i^* maps $\Phi^p(X)$ onto $\Phi^p(A)$.

6. Relative groups

A pair (X, A) is defined as a space X and a subspace A . A function f from a pair (X, A) to a pair (Y, B) is a function f from X to Y that takes A into B .

Let (X, A) be a pair and let $i: A \rightarrow X$ be the identity map. Then $i^*: \Phi^p(X) \rightarrow \Phi^p(A)$. Define the group $\Phi^p(X, A)$ of p -functions on X which are locally zero on A , by

$$\Phi^p(X, A) = i^{*-1}(\Phi_0^p(A)).$$

(6.1) $\bar{\delta}$ maps $\Phi^p(X, A)$ into $\Phi^{p+1}(X, A)$.

Let $\phi \in \Phi^p(X, A)$ so that $i^*\phi \in \Phi_0^p(A)$. Then $i^*(\bar{\delta}\phi) = \bar{\delta}(i^*\phi) \in \Phi_0^{p+1}(A)$ by (5.5) and (4.2).

The group $\Phi_0^p(X, A)$ of p -functions of X mod A which are locally zero on X mod A , is defined by

$$\Phi_0^p(X, A) = \Phi_0^p(X).$$

The group $\Phi_z^p(X, A)$ of p -cocycles of X mod A , is defined by

$$\Phi_z^p(X, A) = \Phi^p(X, A) \cap \Phi_z^p(X).$$

The group $\Phi_B^p(X, A)$ of p -coboundaries of X mod A , is defined by

$$\begin{aligned} \Phi_B^p(X, A) &= \text{span of } \bar{\delta}(\Phi^{p-1}(X, A)) \text{ and } \Phi_0^p(X, A) & \text{if } p > 0 \\ \Phi_B^0(X, A) &= (0). \end{aligned}$$

Then it is clear that

$$\Phi_0^p(X, A) \subset \Phi_B^p(X, A) \subset \Phi_z^p(X, A) \subset \Phi^p(X, A) \quad \text{for all } p.$$

The p^{th} cohomology group of X mod A , $H^p(X, A)$ is defined to be the factor group

$$H^p(X, A) = \Phi_z^p(X, A) / \Phi_B^p(X, A),$$

and γ will denote the natural homomorphism

$$\gamma: \Phi_z^p(X, A) \rightarrow H^p(X, A).$$

Note that if A is the null set $\emptyset$, then $\Phi^p(X, \emptyset) = \Phi^p(X)$, and the two groups $H^p(X, \emptyset)$ and $H^p(X)$ coincide as do the homomorphisms β and γ .

7. Coboundary operator

If (X, A) is a pair, a homomorphism δ from $H^p(A)$ to $H^{p+1}(X, A)$ will be defined in a "natural manner". This homomorphism is called the *coboundary operator*.

As before, let $i: A \rightarrow X$ be the identity map. Let $h \in H^p(A)$. Since i^* is onto by (5.7) and β is onto, there is $\phi \in \Phi^p(X)$ such that $i^*\phi \in \Phi_z^p(A)$ and $\beta i^*\phi = h$.

$$(7.1) \quad \bar{\delta}\phi \in \Phi_z^{p+1}(X, A).$$

$$i^*(\bar{\delta}\phi) = \bar{\delta}(i^*\phi) \in \Phi_0^{p+1}(A) \text{ because } i^*\phi \in \Phi_z^p(A).$$

Hence, $\bar{\delta}\phi \in \Phi_0^{p+1}(X, A)$. Since $\bar{\delta}\bar{\delta}\phi = 0$, $\bar{\delta}\bar{\delta}\phi \in \Phi_0^{p+2}(X, A)$, so $\bar{\delta}\phi \in \Phi_z^{p+1}(X, A)$.

$$(7.2) \quad \text{If } \phi, \phi' \in \Phi^p(X) \text{ such that } \beta i^*\phi = \beta i^*\phi', \text{ then } \gamma\bar{\delta}\phi = \gamma\bar{\delta}\phi'.$$

If $p = 0$, $\beta i^*\phi = \beta i^*\phi'$ implies $i^*\phi = i^*\phi'$ because $\Phi_B^0(A) = (0)$. Then $\phi - \phi' \in \Phi^0(X, A)$ because $i^*(\phi - \phi') = 0$. Hence, $\bar{\delta}(\phi - \phi') \in \Phi_B^1(X, A)$ and $\gamma\bar{\delta}\phi - \gamma\bar{\delta}\phi' = \gamma\bar{\delta}(\phi - \phi') = 0$, so $\gamma\bar{\delta}\phi = \gamma\bar{\delta}\phi'$.

If $p > 0$, $\beta i^*\phi = \beta i^*\phi'$ implies there is $\phi_1 \in \Phi^{p-1}(A)$ such that $i^*\phi - i^*\phi' - \bar{\delta}\phi_1 \in \Phi_0^p(A)$. Let $\bar{\phi}_1 \in \Phi^{p-1}(X)$ such that $i^*\bar{\phi}_1 = \phi_1$. Then $\phi - \phi' - \bar{\delta}\bar{\phi}_1 \in \Phi^p(X, A)$ because

$$i^*(\phi - \phi' - \bar{\delta}\bar{\phi}_1) = i^*\phi - i^*\phi' - i^*\bar{\delta}\bar{\phi}_1 = i^*\phi - i^*\phi' - \bar{\delta}\phi_1 \in \Phi_0^p(A).$$

Then $\gamma\bar{\delta}(\phi - \phi' - \bar{\delta}\bar{\phi}_1) = 0$, so $\gamma\bar{\delta}\phi - \gamma\bar{\delta}\phi' - \gamma\bar{\delta}\bar{\delta}\bar{\phi}_1 = \gamma\bar{\delta}\phi - \gamma\bar{\delta}\phi' = 0$ completing the proof.

Therefore, we can define a homomorphism

$$\delta: H^p(A) \rightarrow H^{p+1}(X, A)$$

such that $\delta h = \gamma\bar{\delta}\phi$ where $\beta i^*\phi = h$.

8. Induced homomorphisms of groups

Let f be a continuous map of the pair (X, A) into the pair (Y, B) . Let i_1 denote the identity map of A into X and i_2 , the identity map of B into Y . Then we have the following diagram of homomorphisms.⁴

$$\begin{array}{ccc} \Phi^p(Y) & \xrightarrow{f^*} & \Phi^p(X) \\ i_2^* \downarrow & & \downarrow i_1^* \\ \Phi^p(B) & \xrightarrow{(f|A)^*} & \Phi^p(A) \end{array}$$

Since $f i_1 = i_2(f|A)$, it follows from (5.4) that $(f|A)^* i_2^* = i_1^* f^*$.

$$(8.1) \quad f^* \text{ maps } \Phi^p(Y, B) \text{ into } \Phi^p(X, A).$$

If $\phi \in \Phi^p(Y, B)$, $i_2^*\phi \in \Phi_0^p(B)$. Then $i_1^*(f^*\phi) = (f|A)^*(i_2^*\phi) \in \Phi_0^p(A)$ by (5.4) and (5.6). Hence, $f^*\phi \in \Phi^p(X, A)$.

$$(8.2) \quad f^* \text{ maps } \Phi_z^p(Y, B) \text{ into } \Phi_z^p(X, A).$$

⁴ $f|A$ denotes the map of A into B obtained by restricting the domain of f to A and the range of f to B .

If $\phi \in \Phi_z^p(Y, B)$, $\bar{\delta}\phi \in \Phi_0^{p+1}(Y, B) = \Phi_0^{p+1}(Y)$. Then $\bar{\delta}(f^*\phi) = f^*(\bar{\delta}\phi) \in \Phi_0^{p+1}(X) = \Phi_0^{p+1}(X, A)$ by (5.5) and (5.6).

(8.3) f^* maps $\Phi_B^p(Y, B)$ into $\Phi_B^p(X, A)$.

If $p = 0$, $\Phi_B^0(Y, B) = (0)$, and the statement is trivial. If $p > 0$ and $\phi \in \Phi_B^p(Y, B)$, there is $\phi' \in \Phi^{p-1}(Y, B)$ such that $\phi - \bar{\delta}\phi' \in \Phi_0^p(Y)$. Then $f^*\phi' \in \Phi^{p-1}(X, A)$ by (8.1), and $f^*\phi - \bar{\delta}f^*\phi' = f^*\phi - f^*\bar{\delta}\phi' = f^*(\phi - \bar{\delta}\phi') \in \Phi_0^p(X)$ by (5.5) and (5.6). Therefore, $f^*\phi \in \Phi_B^p(X, A)$.

As a consequence of (8.2) and (8.3), f^* induces a homomorphism

$$f^*: H^p(Y, B) \rightarrow H^p(X, A)$$

satisfying $f^*\gamma\phi = \gamma f^*\phi$ for $\phi \in \Phi_z^p(Y, B)$.

(8.4) If $f: (X, A) \rightarrow (X, A)$ is the identity, then f^* is the identity isomorphism of $H^p(X, A)$ on itself.

If $\phi \in \Phi_z^p(X, A)$, then $f^*\gamma\phi = \gamma f^*\phi = \gamma\phi$ by (5.3).

(8.5) If $f: (X, A) \rightarrow (Y, B)$ and $g: (Y, B) \rightarrow (Z, C)$, then $(gf)^* = f^*g^*$.

If $\phi \in \Phi_z^p(Z, C)$, then by (5.4)

$$(gf)^*\gamma\phi = \gamma(gf)^*\phi = \gamma f^*g^*\phi = f^*\gamma g^*\phi = f^*g^*(\gamma\phi).$$

(8.6) If $f: (X, A) \rightarrow (Y, B)$, then $f^*\delta = \delta(f|A)^*$.

If $\phi \in \Phi^p(Y)$ such that $\beta i_2^*\phi = h \in H^p(Y)$, then $f^*\delta(h) = f^*\delta(\beta i_2^*\phi) = f^*(\gamma\bar{\delta}\phi) = \gamma(f^*\bar{\delta}\phi) = \gamma\bar{\delta}(f^*\phi)$. Since $\beta i_1^*(f^*\phi) = \beta(f|A)^*i_2^*\phi = (f|A)^*(\beta i_2^*\phi) = (f|A)^*h$, then $\delta((f|A)^*h) = \gamma\bar{\delta}(f^*\phi) = f^*\delta h$ completing the proof.

9. A fundamental lemma

A collection $\mathfrak{U}$ of subsets U of X is called a *covering* of X if $\bigcup_{U \in \mathfrak{U}} U = X$. If $\mathfrak{U}$ and $\mathfrak{B}$ are two coverings of X such that each element of $\mathfrak{U}$ is contained in some element of $\mathfrak{B}$, then $\mathfrak{U}$ is said to be a *refinement* of $\mathfrak{B}$, and this is denoted by $\mathfrak{U} > \mathfrak{B}$. If $U \in \mathfrak{U}$, the *star* of U with respect to $\mathfrak{U}$ is the union of those elements of $\mathfrak{U}$ that meet U . If for each $U \in \mathfrak{U}$, the star of U with respect to $\mathfrak{U}$ is contained in some element of $\mathfrak{B}$, then $\mathfrak{U}$ is said to be a *star refinement* of $\mathfrak{B}$.

If f is a function (not necessarily continuous) from X to Y and if $\mathfrak{B}$ is a covering of Y , $f^{-1}(\mathfrak{B})$ will denote the covering of X consisting of inverse images of elements of $\mathfrak{B}$.

If $\mathfrak{U}$ is an open covering of X (that is, each element of $\mathfrak{U}$ is an open set in X), then $N_{p+1}(\mathfrak{U})$ will be that neighborhood of ΔX^{p+1} defined by $N_{p+1}(\mathfrak{U}) = \bigcup_{U \in \mathfrak{U}} U^{p+1}$. It is obvious that if N is any neighborhood of ΔX^{p+1} , there is an open covering $\mathfrak{U}$ of X such that $N_{p+1}(\mathfrak{U}) \subset N$. If $\mathfrak{U} > \mathfrak{B}$, $N_{p+1}(\mathfrak{U}) \subset N_{p+1}(\mathfrak{B})$.

LEMMA 9.1. Let f, g be two functions (not necessarily continuous) from (X, A) to (Y, B) and let $\phi \in \Phi_z^p(Y, B)$ and $\mathfrak{B}$ be an open covering of Y such that $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{B})$ and $\phi = 0$ on $N_{p+1}(\mathfrak{B}) \cap B^{p+1}$. If there is an open covering $\mathfrak{U}$ of X such

that for each $U \in \mathfrak{U}$ there is a $V \in \mathfrak{B}$ such that $f(U) \cup g(U) \subset V$, then $f^*\phi$ and $g^*\phi$ belong to $\Phi_z^p(X, A)$ and $\gamma f^*\phi = \gamma g^*\phi$.

PROOF: Since $f(U) \cup g(U) \subset V$, it follows that $\mathfrak{U} > f^{-1}(\mathfrak{B})$ and $\mathfrak{U} > g^{-1}(\mathfrak{B})$ so that $f^*\phi$ and $g^*\phi$ vanish on $N_{p+1}(\mathfrak{U}) \cap A^{p+1}$, so $f^*\phi$ and $g^*\phi$ belong to $\Phi^p(X, A)$. Also if $x^{p+2} \in N_{p+2}(\mathfrak{U})$, $\bar{\delta}(f^*\phi)(x^{p+2}) = \bar{\delta}(\phi(f(x^{p+2}))) = 0$ because $f(x^{p+2}) \in N_{p+2}(\mathfrak{B})$. Therefore, $\bar{\delta}(f^*\phi) = 0$ on $N_{p+2}(\mathfrak{U})$, so $f^*\phi \in \Phi_z^p(X, A)$. Similarly $\bar{\delta}(g^*\phi) = 0$ on $N_{p+2}(\mathfrak{U})$, and $g^*\phi \in \Phi_z^p(X, A)$.

If $p > 0$, the deformation cochain $D\phi \in \Phi^{p-1}(X, A)$ is defined by

$$D\phi(x_0, \dots, x_{p-1}) = \sum_{i=0}^{p-1} (-1)^i \phi(g(x_0), \dots, g(x_i), f(x_i), \dots, f(x_{p-1})).$$

That $D\phi \in \Phi^{p-1}(X, A)$ follows from the fact that if

$$(x_0, \dots, x_{p-1}) \in N_p(\mathfrak{U}) \cap A^p, g(x_0), \dots, g(x_{p-1}), f(x_0), \dots, f(x_{p-1})$$

all belong to some element of $\mathfrak{B}$. Therefore, $D\phi$ vanishes on $N_p(\mathfrak{U}) \cap A^p$.

If $p \geq 0$, $D(\bar{\delta}\phi) \in \Phi^p(X, A)$ is defined similarly.

If $p = 0$, for $x \in X$ we have

$D\bar{\delta}\phi(x) = \bar{\delta}\phi(g(x), f(x)) = \phi(f(x)) - \phi(g(x)) = f^*\phi(x) - g^*\phi(x)$. Since $(g(x), f(x)) \in N_1(\mathfrak{B})$ for all $x \in X$, $\bar{\delta}\phi(g(x), f(x)) = 0$. Therefore, $f^*\phi(x) = g^*\phi(x)$ for all x , so $\gamma f^*\phi = \gamma g^*\phi$.

If $p > 0$, direct calculation shows that

$$\bar{\delta}D\phi = f^*\phi - g^*\phi - D\bar{\delta}\phi.$$

If $(x_0, \dots, x_p) \in N_{p+1}(\mathfrak{U})$, then $g(x_0), \dots, g(x_p), f(x_0), \dots, f(x_p)$ all belong to some element of $\mathfrak{B}$. Since $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{B})$,

$$\bar{\delta}D\phi = f^*\phi - g^*\phi \quad \text{on } N_{p+1}(\mathfrak{U}).$$

Then $\gamma f^*\phi - \gamma g^*\phi = \gamma \bar{\delta}D\phi = 0$ completing the proof.

II. VERIFICATION OF THE AXIOMS

10. The axioms

Eilenberg and Steenrod have axiomatised the concepts of homology and cohomology groups of a pair (X, A) (see [4]⁵). It is a consequence of their uniqueness theorem that any cohomology theory satisfying their axioms on a class of pairs which includes the class of all pairs (K, L) where K is a finite complex and L is a closed subcomplex of K agrees on such pairs (K, L) with the cohomology groups based on the cellular structure of the complex. In this section, the axioms for cohomology will be verified thus justifying the term "cohomology theory" for the groups and mappings defined in Section I.

The following definitions will be needed. Two continuous mappings

$$f: (X, A) \rightarrow (Y, B), \quad g: (X, A) \rightarrow (Y, B)$$

⁵ A more detailed discussion of the axiomatic homology theory is being prepared in the form of a book by Eilenberg and Steenrod.

are said to be homotopic if there is a homotopy of f and g in the usual sense which maps A into B at each stage. A sequence of groups and homomorphisms

$$G_1 \xrightarrow{h_1} G_2 \xrightarrow{h_2} \dots \xrightarrow{h_{p-1}} G_p \xrightarrow{h_p} G_{p+1} \xrightarrow{h_{p+1}} \dots$$

is said to be an exact sequence if the kernel of h_{p+1} equals the image of h_p for all $p \geq 1$ and if the kernel of $h_1 = (0)$.

A cohomology theory defined on a class of pairs (X, A) and continuous maps f is a function $H^p(X, A)$ defined for every pair (X, A) in the class and integer $p \geq 0$ with values in the class of abelian groups such that:

(1) If $f: (X, A) \rightarrow (Y, B)$ is a continuous map in the class, there is an attached homomorphism

$$f^*: H^p(Y, B) \rightarrow H^p(X, A)$$

called the induced homomorphism.

(2) For each (X, A) and $p \geq 0$, there is a homomorphism

$$\delta: H^p(A) \rightarrow H^{p+1}(X, A)$$

called the coboundary operator.

These concepts satisfy the following axioms.

AXIOM 1. If f is the identity map of (X, A) on itself, then f^* is the identity isomorphism of $H^p(X, A)$ on itself.

AXIOM 2. If $f: (X, A) \rightarrow (Y, B)$ and $g: (Y, B) \rightarrow (Z, C)$, then $(gf)^* = f^*g^*$.

AXIOM 3. If $f: (X, A) \rightarrow (Y, B)$, then $f^*\delta = \delta(f|A)^*$.

AXIOM 4. (Homotopy Axiom) If X is a compact Hausdorff space and A is a closed subset of X and

$$f: (X, A) \rightarrow (Y, B), \quad g: (X, A) \rightarrow (Y, B)$$

are homotopic, then $f^* = g^*$.

AXIOM 5. (Exactness Axiom) Given (X, A) and the identity mappings

$$i: (A, \emptyset) \rightarrow (X, \emptyset), \quad j: (X, \emptyset) \rightarrow (X, A)$$

the groups and homomorphisms

$$H^0(X, A) \xrightarrow{j^*} H^0(X) \rightarrow \dots \xrightarrow{j^*} H^p(X) \xrightarrow{i^*} H^p(A) \xrightarrow{\delta} H^{p+1}(X, A) \xrightarrow{j^*} \dots$$

form an exact sequence⁶.

AXIOM 6. (Excision Axiom) If U is an open set whose closure is contained in the interior of A , then the identity map $j: (X - U, A - U) \rightarrow (X, A)$ induces isomorphisms

$$j^*: H^p(X, A) \rightarrow H^p(X - U, A - U) \text{ for each } p.$$

AXIOM 7. (Dimension Axiom) If P is a space consisting of a single point, then $H^p(P) = (0)$ for $p \geq 1$.

⁶ This sequence of groups and homomorphisms is called the cohomology sequence of the pair (X, A) .

11. Axioms 1-3.

The cohomology groups, induced homomorphisms, and coboundary operators defined in Section I have been defined on the class of all pairs (X, A) and continuous maps f .

Axioms 1-3, called the algebraic axioms, have been proved in §8, namely (8.4), (8.5) and (8.6).

12. Proof of the homotopy axiom

The homotopy axiom will be proved under the assumption that X is compact Hausdorff but A need not be closed in X .

Let I denote the closed interval from 0 to 1 with its usual topology. Define

$$h_t: (X, A) \rightarrow (X \times I, A \times I) \text{ for each } 0 \leq t \leq 1$$

by

$$h_t(x) = (x, t).$$

Let $F: (X \times I, A \times I) \rightarrow (Y, B)$ be the mapping whose existence is guaranteed by the homotopy of f and g . Then $f = Fh_0$ and $g = Fh_1$. Therefore, $f^* = h_0^*F^*$ and $g^* = h_1^*F^*$, so to show $f^* = g^*$, it suffices to show that $h_0^* = h_1^*$.

Let $\phi \in \Phi_z^p(X \times I, A \times I)$. We shall show that for any $t \in I$, there is $\epsilon_t > 0$ such that

$$\gamma h_t^* \phi = \gamma h_{t'}^* \phi \text{ if } |t - t'| < \epsilon_t.$$

Since $\phi \in \Phi_z^p(X \times I, A \times I)$, there is a covering $\mathfrak{B}$ of $X \times I$ such that $\phi = 0$ on $N_{p+1}(\mathfrak{B}) \cap (A \times I)^{p+1}$ and $\delta\phi = 0$ on $N_{p+2}(\mathfrak{B})$. Let t be a fixed point of I . For each $x \in X$, let U_x be a neighborhood of x and V_x a neighborhood of t such that $U_x \times V_x \subset V$ for some $V \in \mathfrak{B}$. Since X is compact, a finite number of U_x 's, say $U_1, \dots, U_k$ covers X . Let $V_1, \dots, V_k$ be the corresponding V_x 's, so that for all $1 \leq i \leq k$, $U_i \times V_i \subset V$ for some $V \in \mathfrak{B}$. Let $\mathfrak{U}$ be the covering of X whose elements are the U_i ($1 \leq i \leq k$) and choose ϵ_t so that the sphere about t with radius ϵ_t is contained in V_i for all $1 \leq i \leq k$. Let t' be chosen so that $|t - t'| < \epsilon_t$. Then $h_t, h_{t'}$ are two maps of (X, A) into $(X \times I, A \times I)$ such that for each $U \in \mathfrak{U}$ there is a $V \in \mathfrak{B}$ such that $h_t(U) \cup h_{t'}(U) \subset V$. Hence, by Lemma 9.1, $\gamma h_t^* \phi = \gamma h_{t'}^* \phi$.

Since I is connected, this implies that $h_0^*(\gamma\phi) = h_1^*(\gamma\phi)$. Since this is true for all $\phi \in \Phi_z^p(X \times I, A \times I)$, $h_0^* = h_1^*$ completing the proof.

13. Proof of the exactness axiom

Let (X, A) be a pair and

$$i: (A, \phi) \rightarrow (X, \phi), \quad j: (X, \phi) \rightarrow (X, A)$$

be identity mappings. Then the cohomology sequence of (X, A)

$$H^0(X, A) \xrightarrow{j^*} H^0(X) \xrightarrow{i^*} \dots \xrightarrow{i^*} H^{p-1}(A) \xrightarrow{\delta} H^p(X, A) \xrightarrow{j^*} H^p(X) \xrightarrow{i^*} \dots$$

is exact.

The proof is divided into seven parts. Instead of working with the elements of the cohomology groups, the proof is carried out in terms of the cocycles. This is sufficient because the natural homomorphism maps the group of cocycles onto the cohomology group.

(1) Kernel of $i^* \supset$ image of j^* .

Let ϕ be a p -cocycle of $X \bmod A$. Then $i^*j^*\phi = \phi \mid (A^{p+1})$ is locally zero on A from the definition of the p -functions of $X \bmod A$. Hence, $i^*j^*\phi \in \Phi_0^p(A) \subset \Phi_B^p(A)$, so $i^*j^*(\gamma\phi) = \beta(i^*j^*\phi) = 0$.

(2) Kernel of $i^* \subset$ image of j^* .

Let ϕ be a p -cocycle on X such that $i^*\phi$ is a p -coboundary on A . Then there is $\phi' \in \Phi^{p-1}(A)$ such that $i^*\phi - \bar{\delta}\phi' \in \Phi_0^p(A)$. By (5.7), there is $\phi'' \in \Phi^{p-1}(X)$ such that $i^*\phi'' = \phi'$. We will show that $\phi - \bar{\delta}\phi''$ is a p -cocycle of $X \bmod A$. $\bar{\delta}(\phi - \bar{\delta}\phi'') = \bar{\delta}\phi \in \Phi_0^{p+1}(X)$ because ϕ is a cocycle on X , so $\phi - \bar{\delta}\phi''$ is a p -cocycle. Moreover

$$i^*(\phi - \bar{\delta}\phi'') = i^*\phi - \bar{\delta}i^*\phi'' = i^*\phi - \bar{\delta}\phi' \in \Phi_0^p(A), \text{ so } \phi - \bar{\delta}\phi'' \in \Phi^p(X, A),$$

and $\phi - \bar{\delta}\phi''$ has been shown to be a p -cocycle of $X \bmod A$. But $\phi - j^*(\phi - \bar{\delta}\phi'') = \bar{\delta}\phi'' \in \Phi_B^p(X)$ showing that $\beta\phi = \beta j^*(\phi - \bar{\delta}\phi'') = j^*\gamma(\phi - \bar{\delta}\phi'')$, so the kernel of $i^* \subset$ image of j^* .

(3) Kernel of $j^* \supset$ image of δ .

Let ϕ be a $(p-1)$ -cocycle of A . Then $j^*\delta(\beta\phi) = j^*\gamma(\bar{\delta}\phi')$ where ϕ' is a $(p-1)$ -function of X such that $i^*\phi' = \phi$. Then $j^*\gamma(\bar{\delta}\phi') = \beta j^*(\bar{\delta}\phi') = \beta(\bar{\delta}j^*\phi') = \beta\bar{\delta}\phi' = 0$.

(4) Kernel of $j^* \subset$ image of δ .

Let ϕ be a p -cocycle of $X \bmod A$ such that $j^*\phi$ is a coboundary on X . Then there is $\phi' \in \Phi^{p-1}(X)$ such that $j^*\phi - \bar{\delta}\phi' \in \Phi_0^p(X)$. Then $i^*(j^*\phi - \bar{\delta}\phi') \in \Phi_0^p(A)$ and $i^*(j^*\phi) \in \Phi_0^p(A)$. Hence, $i^*(j^*\phi) - i^*(j^*\phi - \bar{\delta}\phi') = i^*\bar{\delta}\phi' = \bar{\delta}i^*\phi' \in \Phi_0^p(A)$. Therefore, $i^*\phi'$ is a cocycle on A , and $\delta(\beta i^*\phi') = \gamma\bar{\delta}\phi'$. Since $j^*(\phi - \bar{\delta}\phi') \in \Phi_0^p(X)$, $\phi - \bar{\delta}\phi' \in \Phi_0^p(X, A)$, so $\gamma\phi = \gamma\bar{\delta}\phi' = \delta(\beta i^*\phi')$ showing that the kernel of $j^* \subset$ image of δ .

(5) Kernel of $\delta \supset$ image of i^* .

Let ϕ be a $(p-1)$ -cocycle on X . Then $\delta i^*(\beta\phi) = \delta(\beta i^*\phi) = \gamma\bar{\delta}\phi = 0$ because $\bar{\delta}\phi$ is locally zero on X since ϕ is a cocycle on X .

(6) Kernel of $\delta \subset$ image of i^* .

Let ϕ be a $(p-1)$ -cocycle on A such that $\delta(\beta\phi) = 0$. Then there is $\phi' \in \Phi^{p-1}(X)$ such that $i^*\phi' = \phi$ so that $\delta(\beta\phi) = \delta(\beta i^*\phi') = \gamma\bar{\delta}\phi' = 0$. Hence, there is $\phi'' \in \Phi^{p-1}(X, A)$ such that $\bar{\delta}\phi' - \bar{\delta}\phi'' \in \Phi_0^p(X, A)$. Then $\phi' - j^*\phi''$ is a p -cocycle on X such that $i^*(\phi' - j^*\phi'') = i^*\phi' = \phi$, so $i^*(\beta(\phi' - j^*\phi'')) = \beta\phi$ showing that the kernel of $\delta \subset$ image of i^* .

(7) j^* maps $H^0(X, A)$ isomorphically into $H^0(X)$.

Let ϕ be a 0-cocycle of $X \bmod A$ such that $j^*\phi$ is a coboundary on X . Since the only 0-coboundary on X is the function which is identically zero on X , $j^*\phi = 0$. Then $\phi = 0$ and ϕ is a coboundary of $X \bmod A$ showing that the kernel of $j^* = (0)$ for the dimension zero.

14. Proof of the excision axiom

Let (X, A) be a pair and let U be an open subset of X such that $\bar{U} \subset \text{interior of } A$. Let $j: (X - U, A - U) \rightarrow (X, A)$ be the identity.

It is clear that j^* maps $\Phi^p(X, A)$ onto $\Phi^p(X - U, A - U)$. We shall show that $j^{*-1}(\Phi_0^p(X - U, A - U)) = \Phi_0^p(X, A)$. Let $\phi \in \Phi^p(X, A)$ such that $j^*\phi \in \Phi_0^p(X - U, A - U)$. Then $j^*\phi = 0$ on some neighborhood N of $\Delta(X - U)^{p+1}$. Since j is the identity, $\phi = j^*\phi$ on $(X - U)^{p+1}$, so $\phi = 0$ on N . Since $\phi \in \Phi^p(X, A)$, $\phi = 0$ on a neighborhood N' of ΔA^{p+1} . Now $N \cap (X - \bar{U})^{p+1}$ and $N' \cap (\text{int } A)^{p+1}$ are both open in X^{p+1} . Then $[N \cap (X - \bar{U})^{p+1}] \cup [N' \cap (\text{int } A)^{p+1}]$ is a neighborhood of ΔX^{p+1} on which ϕ vanishes, so $\phi \in \Phi_0^p(X, A)$.

Since $j^*\bar{\delta} = \bar{\delta}j^*$, it follows that j^* maps $\Phi_z^p(X, A)$ onto $\Phi_z^p(X - U, A - U)$ and that $j^{*-1}(\Phi_B^p(X - U, A - U)) = \Phi_B^p(X, A)$. Therefore, j^* is an isomorphism of $H^p(X, A)$ onto $H^p(X - U, A - U)$ proving the excision axiom.

NOTE. A stronger form of excision is proved in §18 for compact pairs (X, A) .

15. The dimension axiom

Let P be a space consisting of a single point x . Then $\pi_i = \pi_j$ for all i, j and any $p \geq 1$.

(1) If p is odd,

$$\bar{\delta}\phi(x^{p+2}) = \sum_{i=0}^{p+2} (-1)^i \phi(\pi_i(x^{p+2})) = \phi\pi_0(x^{p+2}).$$

Hence, if $\phi \in \Phi_z^p(P)$, $\phi\pi_0(x^{p+2}) = 0$, so $\phi = 0$. Then $\Phi_z^p(P) = (0)$, so $H^p(P) = (0)$.

(2) If p is even and $p > 0$,

$\bar{\delta}\phi(x^{p+2}) = 0$ for all ϕ , so that $\Phi_z^p(P) = \Phi^p(P)$. For each $\phi \in \Phi^p(P)$, define $\phi' \in \Phi^{p-1}(P)$ by $\phi'(x^p) = \phi(x^{p+1})$. Then $\bar{\delta}\phi'(x^{p+1}) = \phi'(\pi_0(x^{p+1})) = \phi'(x^p) = \phi(x^{p+1})$ so that $\Phi_B^p(P) = \Phi^p(P)$, and $H^p(P) = (0)$.

If $p = 0$, $\Phi_z^0(P) = \Phi^0(P)$ and $\Phi_B^0(0) = (0)$. Then $H^0(P) = \Phi^0(P)$. If $\phi \in \Phi^0(P)$, the mapping $\phi \rightarrow \phi(x)$ induces the isomorphism $H^0(P) \approx G$ showing that the cohomology theory is non-trivial if the coefficient group is non-trivial.

16. Cochains

The cohomology groups defined in Section I could have been obtained from a cochain complex in the following way. Since $\Phi_0^p(X, A)$ is a subgroup of $\Phi^p(X, A)$, the group of p -cochains of $X \bmod A$, $C^p(X, A)$ can be defined to be the factor group

$$C^p(X, A) = \Phi^p(X, A) / \Phi_0^p(X, A).$$

If $\phi \in \Phi^p(X, A)$, the cochain corresponding to it is denoted by $\{\phi\}$. Then $\bar{\delta}: \Phi^p(X, A) \rightarrow \Phi^{p+1}(X, A)$ induces a homomorphism

$$\delta^*: C^p(X, A) \rightarrow C^{p+1}(X, A)$$

satisfying $\delta^*\{\phi\} = \{\bar{\delta}\phi\}$ for $\phi \in \Phi^p(X, A)$.

$$(16.1) \quad \delta^*\delta^* = 0.$$

This follows immediately from (4.1).

Therefore, if $Z^p(X, A)$ is defined to be the kernel of δ^* in $C^p(X, A)$ and $B^p(X, A)$ is defined to be the image of $C^{p-1}(X, A)$ under δ^* for $p > 0$ and $B^0(X, A)$ is defined to be (0) , we see that

$$B^p(X, A) \subset Z^p(X, A) \subset C^p(X, A) \text{ for all } p \geq 0.$$

Then the cohomology group can be defined to be the factor group

$$H^p(X, A) = Z^p(X, A)/B^p(X, A).$$

The induced homomorphisms can be defined in terms of the cochains, and, in fact, the whole theory could have been constructed from this viewpoint. However, since the proof of most propositions would ultimately have to be carried out in terms of the functions, it seems preferably to define the cohomology groups directly in terms of the functions instead of the cochains.

NOTE. Using cochains the proof of the exactness axiom follows immediately from a general theorem of Kelley-Pitcher [5].

III. COMPARISONS WITH OTHER THEORIES

17. The continuity axiom

Let $\{X_\alpha: \pi_{\alpha\beta}\}$ be an inverse system of compact Hausdorff spaces [6]. For each α , let A_α be a closed subset of X_α such that if $\beta > \alpha$, then $\pi_{\beta\alpha}(A_\beta) \subset A_\alpha$. Then we say that $\{(X_\alpha, A_\alpha); \pi_{\alpha\beta}\}$ is an *inverse system of pairs*, and if $X = \varprojlim X_\alpha$ and $A = \varprojlim A_\alpha$, we write $(X, A) = \varprojlim (X_\alpha, A_\alpha)$.

Let $\pi_\alpha: (X, A) \rightarrow (X_\alpha, A_\alpha)$ be the continuous map defined by $\pi_\alpha\{x_\beta\} = x_\alpha$ (projection onto α^{th} coordinate). Then if $\beta > \alpha$, $\pi_\alpha = \pi_{\beta\alpha}\pi_\beta$, so $\pi_\beta^*\pi_{\beta\alpha}^* = \pi_\alpha^*$. Let $H^p(X, A)$ be a cohomology theory. Define $\bar{H}^p(X, A) = \varprojlim H^p(X_\alpha, A_\alpha)$. If $h_\alpha \in H^p(X_\alpha, A_\alpha)$, let $\{h_\alpha\}$ be the element of $\bar{H}^p(X, A)$ determined by it. Define $\pi^*: \bar{H}^p(X, A) \rightarrow H^p(X, A)$ by $\pi^*\{h_\alpha\} = \pi_\alpha^*h_\alpha$.

CONTINUITY AXIOM. If $(X, A) = \varprojlim (X_\alpha, A_\alpha)$ and $\bar{H}^p(X, A)$ and π^* are as above, then π^* is an isomorphism of $\bar{H}^p(X, A)$ onto $H^p(X, A)$ for $p \geq 0$.

A cohomology theory which satisfies the continuity axiom in addition to the axioms stated in §10 is unique on the set of pairs (X, A) with X compact Hausdorff and A closed in X . This follows from the proposition⁷ that such a pair (X, A) is expressible as an inverse limit of pairs (K, L) where K is a finite complex and L is a closed subcomplex of K ; for the continuity axiom states that the groups of (X, A) are determined by the groups of (K, L) , which are unique for theories satisfying the other axioms. Steenrod [7] has shown that the Čech cohomology theory satisfies the continuity axiom, hence after we have shown that the group constructed in Section I also satisfies this axiom, it will follow that the new theory agrees with the Čech theory for compact pairs.

THEOREM 17.1. (*Strong Excision Axiom*) If (X, A) is a pair with X compact

⁷ This proposition is an unpublished result of Eilenberg to appear in the forthcoming book on axiomatic homology theory by Eilenberg and Steenrod.

Hausdorff and A closed in X and H^p is a cohomology theory satisfying the continuity axiom, then the identity map

$$j: (X - \text{int } A, A - \text{int } A) \rightarrow (X, A)$$

induces isomorphisms

$$j^*: H^p(X, A) \approx H^p(X - \text{int } A, A - \text{int } A) \text{ for all } p.$$

PROOF: Let $\{U_\alpha\}$ be the family of open subsets of X such that $\bar{U}_\alpha \subset \text{int } A$, indexed by a set of indices $\{\alpha\}$. Define $\alpha > \beta$ if $U_\alpha \supset U_\beta$. For $\alpha > \beta$, let $\pi_{\alpha\beta}: (X - U_\alpha, A - U_\alpha) \rightarrow (X - U_\beta, A - U_\beta)$ denote the identity map. Let $j_\alpha: (X - U_\alpha, A - U_\alpha) \rightarrow (X, A)$ be the identity. Then by the excision axiom (axiom 6 for the cohomology groups), j_α induces isomorphisms

$$j_\alpha^*: H^p(X, A) \approx H^p(X - U_\alpha, A - U_\alpha).$$

For $\alpha > \beta$, $j_\alpha = j_\beta \pi_{\alpha\beta}$, so $j_\alpha^* = \pi_{\alpha\beta}^* j_\beta^*$. Therefore, $H^p(X - U_\alpha, A - U_\alpha)$ is isomorphic to $H^p(X, A)$ and $\pi_{\alpha\beta}^*$ is an isomorphism of $H^p(X - U_\beta, A - U_\beta)$ onto $H^p(X - U_\alpha, A - U_\alpha)$.

The family $\{(X - U_\alpha, A - U_\alpha), \pi_{\alpha\beta}\}$ is an inverse system of compact pairs directed by inclusion. For such a system, the limit space is isomorphic to the intersection of the spaces. We shall identify this intersection with the limit space. Since $X - \text{int } A = \bigcap_\alpha (X - U_\alpha)$ and $A - \text{int } A = \bigcap_\alpha (A - U_\alpha)$, we see that $(X - \text{int } A, A - \text{int } A) = \varprojlim (X - U_\alpha, A - U_\alpha)$. Then by the continuity axiom, π^* is an isomorphism of $\varinjlim H^p(X - U_\alpha, A - U_\alpha)$ onto $H^p(X - \text{int } A, A - \text{int } A)$. Since $\varinjlim H^p(X - U_\alpha, A - U_\alpha)$ is isomorphic to $H^p(X - U_\alpha, A - U_\alpha)$, π_α^* is an isomorphism

$$\pi_\alpha^* H^p(X - U_\alpha, A - U_\alpha) \rightarrow H^p(X - \text{int } A, A - \text{int } A)$$

where π_α^* is the homomorphism induced by the identity map

$$\pi_\alpha: (X - \text{int } A, A - \text{int } A) \rightarrow (X - U_\alpha, A - U_\alpha).$$

Then the identity map $j: (X - \text{int } A, A - \text{int } A) \rightarrow (X, A)$ is the composition $j = j_\alpha \pi_\alpha$, so $j^* = \pi_\alpha^* j_\alpha^*$. Since both π_α^* and j_α^* are isomorphisms, it follows that j^* is also an isomorphism.

18. Proof of the continuity axiom

In the following, the notation of §17 will be used. Thus $(X, A) = \varprojlim (X_\alpha, A_\alpha)$ where $\{(X_\alpha, A_\alpha); \pi_{\alpha\beta}\}$ is an inverse system of compact pairs, and $\pi_\alpha: (X, A) \rightarrow (X_\alpha, A_\alpha)$.

LEMMA 18.1. *If $\mathcal{U}_\alpha$ is an open covering of X_α and $\mathcal{U}$ is an open covering of X , then there exists an index $\gamma > \alpha$, an open covering $\mathcal{B}_\gamma$ of X_γ , and a function $f: (X_\gamma, A_\gamma) \rightarrow (X, A)$ (not necessarily continuous) such that:*

- (1) If $V \in \mathcal{B}_\gamma$, $f(V) \subset$ some element of $\mathcal{U}$.
- (2) If $V \in \mathcal{B}_\gamma$, $V \cup \pi_\gamma f(V) \subset$ some element of $\pi_{\gamma\alpha}^{-1}(\mathcal{U}_\alpha)$.
- (3) If $V' \in \pi_\gamma^{-1}(\mathcal{B}_\gamma)$, $V' \cup f\pi_\gamma(V') \subset$ some element of $\mathcal{U}$.

PROOF: It follows from the compactness assumptions that there is $\beta > \alpha$ and a covering $\mathfrak{U}_\beta$ of X_β such that $\pi_\beta^{-1}(\mathfrak{U}_\beta) > \mathfrak{U}$. Let $\mathfrak{B}_\beta$ be a star refinement of both $\mathfrak{U}_\beta$ and $\pi_{\beta\alpha}^{-1}(\mathfrak{U}_\alpha)$. Let M be the union of those elements of $\mathfrak{B}_\beta$ which meet $\pi_\beta(X)$ and N be the union of those elements of $\mathfrak{B}_\beta$ which meet $\pi_\beta(A)$. Since M is a neighborhood of $\pi_\beta(X)$ in X_β and N is a neighborhood of $\pi_\beta(A)$ in X_β , there is $\gamma > \beta$ such that $\pi_{\gamma\beta}(X_\gamma) \subset M$ and $\pi_{\gamma\beta}(A_\gamma) \subset N$. Let $\mathfrak{B}_\gamma$ be the set of non-empty elements of $\pi_{\gamma\beta}^{-1}(\mathfrak{B}_\beta)$. Then every element of $\mathfrak{B}_\gamma$ meets $\pi_\gamma(X)$, and every point of A_γ is contained in some element of $\mathfrak{B}_\gamma$ that meets $\pi_\gamma(A)$. Also $\pi_\gamma^{-1}(\mathfrak{B}_\gamma)$ is a star refinement of $\mathfrak{U}$, and $\mathfrak{B}_\gamma$ is a star refinement of $\pi_{\gamma\alpha}^{-1}(\mathfrak{U}_\alpha)$.

For each $V_k \in \mathfrak{B}_\gamma$ such that $V_k \cap \pi_\gamma(A) \neq \emptyset$ choose $x_k \in A$ such that $\pi_\gamma(x_k) \in V_k$. If $V_j \cap \pi_\gamma(A) = \emptyset$, choose $x_j \in X$ such that $\pi_\gamma(x_j) \in V_j$. If $x \in A_\gamma$, choose V_k containing x such that $V_k \cap \pi_\gamma(A) \neq \emptyset$ and define $f(x) = x_k$. For each $x \in X_\gamma - A_\gamma$, choose V_j containing x and define $f(x) = x_j$. Then f maps (X_γ, A_γ) into (X, A) and if $V \in \mathfrak{B}_\gamma$, $f(V) \subset \text{star of } \pi_\gamma^{-1}(V)$ with respect to the covering $\pi_\gamma^{-1}(\mathfrak{B}_\gamma)$. Since $\pi_\gamma^{-1}(\mathfrak{B}_\gamma)$ is a star refinement of $\mathfrak{U}$, $f(V) \subset$ some element of $\mathfrak{U}$ proving (1).

To prove (2) note that $f(V) \subset \text{star of } \pi_\gamma^{-1}(V)$ with respect to $\pi_\gamma^{-1}(\mathfrak{B}_\gamma)$ implies that $\pi_\gamma f(V) \subset \text{star of } V$ with respect to $\mathfrak{B}_\gamma$. Hence, $V \cup \pi_\gamma f(V) \subset \text{star of } V$ with respect to $\mathfrak{B}_\gamma$. Since $\mathfrak{B}_\gamma$ is a star refinement of $\pi_{\gamma\alpha}^{-1}(\mathfrak{U}_\alpha)$, $V \cup \pi_\gamma f(V) \subset$ some element of $\pi_{\gamma\alpha}^{-1}(\mathfrak{U}_\alpha)$.

To prove (3), let $V' \in \pi_\gamma^{-1}(\mathfrak{B}_\gamma)$ and let $\pi_\gamma(V') = V$. Then $f\pi_\gamma(V') = f(V) \subset \text{star of } \pi_\gamma^{-1}(V)$ with respect to $\pi_\gamma^{-1}(\mathfrak{B}_\gamma) = \text{star of } V'$ with respect to $\pi_\gamma^{-1}(\mathfrak{B}_\gamma)$. Hence, $V' \cup f\pi_\gamma(V') \subset \text{star of } V'$ with respect to $\pi_\gamma^{-1}(\mathfrak{B}_\gamma) \subset$ some element of $\mathfrak{U}$ because $\pi_\gamma^{-1}(\mathfrak{B}_\gamma)$ is a star refinement of $\mathfrak{U}$.

PROOF OF THE CONTINUITY AXIOM:

(a) π^* is onto.

Let $h \in H^p(X, A)$ and choose $\phi \in \Phi_z^p(X, A)$ such that $\gamma\phi = h$. Let $\mathfrak{U}$ be an open covering of X such that $\phi = 0$ on $N_{p+1}(\mathfrak{U}) \cap A^{p+1}$ and $\delta\phi = 0$ on $N_{p+2}(\mathfrak{U})$. Let X_γ , $\mathfrak{B}_\gamma$ and f be chosen with respect to $\mathfrak{U}$ as in Lemma 18.1 (X_α and $\mathfrak{U}_\alpha$ may be arbitrary). Then $\delta(f^*\phi) = 0$ on $N_{p+2}(\mathfrak{B}_\gamma)$ because $f(N_{p+2}(\mathfrak{B}_\gamma)) \subset N_{p+2}(\mathfrak{U})$. Also $f^*\phi = 0$ on $N_{p+1}(\mathfrak{B}_\gamma) \cap A_\gamma$. Therefore, $f^*\phi \in \Phi_z^p(X_\gamma, A_\gamma)$. We will show that $\pi^*\{\gamma(f^*\phi)\} = \gamma\phi$. Note that

$$\pi^*\{\gamma(f^*\phi)\} = \pi_\gamma^*\gamma(f^*\phi) = \gamma\pi_\gamma^*f^*\phi = \gamma(f\pi_\gamma)^*\phi.$$

Let $i: (X, A) \rightarrow (X, A)$ be the identity. Then if $V' \in \pi_\gamma^{-1}(\mathfrak{B}_\gamma)$, $i(V') \cup f\pi_\gamma(V') \subset$ some element of $\mathfrak{U}$ by (3) of Lemma 18.1. Hence, by Lemma 9.1, $\gamma(f\pi_\gamma)^*\phi = \gamma i^*\phi = \gamma\phi$ showing that π^* is onto.

(b) Kernel of $\pi^* = (0)$.

Let $\pi^*\{h_\alpha\} = 0$ so that $\pi_\alpha^*h_\alpha = 0$. Choose $\phi_\alpha \in \Phi_z^p(X_\alpha, A_\alpha)$ so that $\gamma\phi_\alpha = h_\alpha$. Then there is a covering $\mathfrak{U}_\alpha$ of X_α such that $\phi_\alpha = 0$ on $N_{p+1}(\mathfrak{U}_\alpha) \cap A_\alpha^{p+1}$ and $\delta\phi_\alpha = 0$ on $N_{p+2}(\mathfrak{U}_\alpha)$. Since $\pi_\alpha^*\gamma\phi_\alpha = \gamma\pi_\alpha^*\phi_\alpha = 0$, there is a covering $\mathfrak{U}$ of X and $\phi' \in \Phi^{p-1}(X, A)$ such that $\pi_\alpha^*\phi_\alpha = \delta\phi'$ on $N_{p+1}(\mathfrak{U})$ and $\phi' = 0$ on $N_p(\mathfrak{U}) \cap A^p$.

Choose X_γ , $\mathfrak{B}_\gamma$ and f as in Lemma 18.1. Then $\bar{\delta}(f^* \phi') = f^* \pi_\alpha^* \phi_\alpha = (\pi_\alpha f)^* \phi_\alpha$ on $N_{p+1}(\mathfrak{B}_\gamma)$. If $V \in \mathfrak{B}_\gamma$, $V \cup \pi_\gamma f(V) \subset$ some element of $\pi_{\gamma\alpha}^{-1}(\mathfrak{U}_\alpha)$ by (2) of Lemma 18.1, so that $\pi_{\gamma\alpha}(V) \cup \pi_{\gamma\alpha}\pi_\gamma f(V)$ is contained in some element of $\mathfrak{U}_\alpha$. By Lemma 9.1, $\gamma\pi_{\gamma\alpha}^* \phi_\alpha = \gamma(\pi_{\gamma\alpha}\pi_\gamma f)^* \phi_\alpha = \gamma(\pi_\alpha f)^* \phi_\alpha = 0$. Hence, $\pi_{\gamma\alpha}^* h_\alpha = \pi_{\gamma\alpha}^* \gamma \phi_\alpha = \gamma\pi_{\gamma\alpha}^* \phi_\alpha = 0$, so $\{h_\alpha\} = 0$ showing that the kernel of $\pi^* = (0)$.

REMARK. It follows from Theorem 17.1 that the cohomology theory defined in Section I satisfies the strong form of the excision axiom for compact pairs (X, A) .

19. Groups of an infinite complex

Let K be a locally finite simplicial complex and let L be a closed subcomplex of K . Let X be the space of K and A be the space of L . $C^p(K, L)$ will denote the group of infinite p -cochains of K modulo L . ($f \in C^p(K, L)$) means that $f(v_0, \dots, v_p) \in G$ is a function of ordered $(p+1)$ -tuples of vertices v_i such that the $(p+1)$ -vertices $v_0, \dots, v_p$ are contained in a simplex of K . If they are in a simplex of L , $f = 0$. $H^p(K, L)$ will denote the p^{th} cohomology group of K mod L based on infinite cochains. Let ψ denote the natural homomorphism of the p -cocycles of K mod L , $Z^p(K, L)$, onto $H^p(K, L)$.

Let $\mathfrak{U}(K)$ be the covering of X consisting of the point set stars of the vertices of the first barycentric subdivision K' of K . For each $x \in X$, choose a vertex v_x of K whose closed star in K' contains x . Then if $(x_0, \dots, x_p) \in N_{p+1}(\mathfrak{U}(K))$, $v_{x_0}, \dots, v_{x_p}$ is an ordered collection of vertices all lying on some simplex of K and if $(x_0, \dots, x_p) \in N_{p+1}(\mathfrak{U}(K)) \cap A^{p+1}$, then $v_{x_0}, \dots, v_{x_p}$ all lie on a simplex of L .

Define a homomorphism

$$\tau: C^p(K, L) \rightarrow \Phi^p(X, A)$$

$$\text{by } \tau c(x_0, \dots, x_p) = \begin{cases} c(v_{x_0}, \dots, v_{x_p}) & \text{if } (x_0, \dots, x_p) \in N_{p+1}(\mathfrak{U}(K)) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{\delta}\tau c = \tau\bar{\delta}c$ on $N_{p+2}(\mathfrak{U}(K))$ and, therefore, τ induces a homomorphism

$$\tau^*: H^p(K, L) \rightarrow H^p(X, A)$$

satisfying $\tau^* \psi c = \gamma \tau c$ for $c \in Z^p(K, L)$.

THEOREM 19.1. τ^* is an isomorphism of $H^p(K, L)$ onto $H^p(X, A)$.

PROOF:

(a) Kernel of $\tau^* = (0)$.

Let $\gamma \tau c = 0$. Then there is $\phi \in \Phi^{p-1}(X, A)$ and an open covering $\mathfrak{U}$ of X such that $\phi = 0$ on $N_p(\mathfrak{U}) \cap A^p$ and $\tau c = \bar{\delta}\phi$ on $N_{p+1}(\mathfrak{U})$. In addition, we can assume that $\mathfrak{U}$ is a refinement of $\mathfrak{U}(K)$ because if it isn't, it can be replaced by a simultaneous refinement of it and $\mathfrak{U}(K)$ which still satisfies the above conditions with respect to ϕ . Let $\sigma(K)$ be a subdivision of K such that each simplex of $\sigma(K)$ is contained completely in some element of $\mathfrak{U}$. Such a subdivision exists because

K is locally finite. Then each cell $(v_0, \dots, v_p)$ of $\sigma(K)$, when considered as a point of X^{p+1} , is an element of $N_{p+1}(\mathfrak{U})$. Let $\bar{\Phi}^p(X, A)$ be that subgroup of $\Phi^p(X, A)$ whose elements vanish on $N_{p+1}(\mathfrak{U}) \cap A^{p+1}$. Then if $\phi' \in \bar{\Phi}(X, A)$, $\bar{\delta}\phi' \in \bar{\Phi}^{p+1}(X, A)$. Also for any $c \in C^p(K, L)$, $\tau c \in \bar{\Phi}^p(X, A)$ because $\mathfrak{U}$ is a refinement of $\mathfrak{U}(K)$. Note that the ϕ considered above is an element of $\bar{\Phi}^{p-1}(X, A)$. For each $\phi' \in \bar{\Phi}^p(X, A)$, define $\lambda\phi' \in C^p(\sigma(K), \sigma(L))$ by

$$\lambda\phi'(v_0, \dots, v_p) = \phi'(v_0, \dots, v_p).$$

Then λ is a homomorphism

$$\lambda: \bar{\Phi}^p(X, A) \rightarrow C^p(\sigma(K), \sigma(L))$$

such that $\lambda\bar{\delta}\phi' = \delta\lambda\phi'$ for any $\phi' \in \bar{\Phi}^p(X, A)$. Then for the $\phi \in \bar{\Phi}^{p-1}(X, A)$ already chosen, $\lambda\tau c = \lambda\bar{\delta}\phi = \delta\lambda\phi$ because $\tau c = \bar{\delta}\phi$ on $N_{p+1}(\mathfrak{U})$. Therefore, $\lambda\tau c \sim 0$ on $\sigma(K) \bmod \sigma(L)$. The mapping $\lambda\tau: C^p(K, L) \rightarrow C^p(\sigma(K), \sigma(L))$ is determined by a simplicial map of $\sigma(K)$ into K which takes each vertex v of $\sigma(K)$ into a vertex $\bar{v}$ of K whose closed star in K' contains v . It is well known that such a map induces isomorphisms of the cohomology groups of $K \bmod L$ with those of $\sigma(K) \bmod \sigma(L)$. Hence, $\lambda\tau c \sim 0$ on $\sigma(K) \bmod \sigma(L)$ implies $c \sim 0$ on $K \bmod L$ showing that the kernel of $\tau^* = (0)$.

(b) τ^* is onto.

Let $\phi \in \Phi_z^p(X, A)$ and choose an open covering $\mathfrak{U} > \mathfrak{U}(K)$ such that $\phi = 0$ on $N_{p+1}(\mathfrak{U}) \cap A^{p+1}$ and $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{U})$. Let $\sigma(K)$ be a subdivision of K such that the covering $\mathfrak{B}$ of X consisting of the point set stars of the vertices of the first barycentric subdivision $\sigma(K)'$ of $\sigma(K)$ is a star refinement of $\mathfrak{U}$. Define

$$\lambda: \bar{\Phi}^p(X, A) \rightarrow C^p(\sigma(K), \sigma(L)) \quad \text{as in (a).}$$

Then since $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{U})$, $\delta(\lambda\phi) = 0$, so $\lambda\phi \in Z^p(\sigma(K), \sigma(L))$. Since $\lambda\tau$ induces isomorphisms of the cohomology groups of $K \bmod L$ onto those of $\sigma(K) \bmod \sigma(L)$, there is $c \in Z^p(K, L)$ and $u \in C^{p-1}(\sigma(K), \sigma(L))$ such that $\lambda\tau c - \lambda\phi = \delta u$. Then $\tau c - \phi \in \Phi_z^p(X, A)$ such that $\tau c - \phi = 0$ on $N_{p+1}(\mathfrak{U}) \cap A^{p+1}$, $\bar{\delta}(\tau c - \phi) = 0$ on $N_{p+2}(\mathfrak{U})$, and $\lambda(\tau c - \phi) = \delta u$.

Define $\tau_\sigma: C^p(\sigma(K), \sigma(L)) \rightarrow \bar{\Phi}^p(X, A)$ in the same way that τ was defined for K (that is, for each $x \in X$, choose a vertex v_x of $\sigma(K)$ whose closed star in $\sigma(K)'$ contains x , etc.). Then $\tau_\sigma\lambda(\tau c - \phi) = \tau_\sigma\delta u = \bar{\delta}\tau_\sigma u$.

Let f be the function from (X, A) to (X, A) defined by $f(x) = v_x$ where $v_x \in \sigma(K)$ was used to define τ_σ , and let $i: (X, A) \rightarrow (X, A)$ be the identity map. Then since $\mathfrak{B}$ is a star refinement of $\mathfrak{U}$, for any $V \in \mathfrak{B}$, there is $U \in \mathfrak{U}$ such that $f(V) \cup i(V) \subset U$. Then by Lemma 9.1, $\gamma f^*(\tau c - \phi) = \gamma i^*(\tau c - \phi) = \gamma\tau c - \gamma\phi$. Since

$$\gamma f^*(\tau c - \phi) = \gamma\tau_\sigma\lambda(\tau c - \phi) = \gamma\bar{\delta}\tau_\sigma u = 0, \text{ then } \gamma\tau c = \gamma\phi$$

showing that $\tau^*(\psi c) = \gamma\phi$, so τ^* is onto.

APPENDIX A

20. Another construction of the cohomology theory

Define a p -simplex of X to be any ordered $(p+1)$ -tuple $(x_0, \dots, x_p)$ of points of X . The points x_i are called the *vertices* of the simplex. Define

$$\partial(x_0, \dots, x_p) = \sum_{i=0}^p (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_p).$$

Then incidence numbers can be defined by collecting terms on the right hand side. The collection of all simplices of all dimensions forms a closure finite complex which will be denoted by K_X . If A is a subset of X , then K_A is a closed subcomplex of K_X .

Let $\mathfrak{U}$ be an arbitrary open covering of X . A p -simplex $(x_0, \dots, x_p)$ of X will be said to be contained in $\mathfrak{U}$ if all its vertices lie in the same element of $\mathfrak{U}$. The set of all simplices contained in $\mathfrak{U}$ forms a closed subcomplex $K_X(\mathfrak{U})$ of K_X . If $\mathfrak{U}_1, \mathfrak{U}_2$ are two open coverings of X with $\mathfrak{U}_1 > \mathfrak{U}_2$, then $K_X(\mathfrak{U}_1) \subset K_X(\mathfrak{U}_2)$, and the identity simplicial map from $K_X(\mathfrak{U}_1)$ to $K_X(\mathfrak{U}_2)$ will be denoted by

$$\pi_1^2 : K_X(\mathfrak{U}_1) \rightarrow K_X(\mathfrak{U}_2).$$

Define $K_A(\mathfrak{U}) = K_X(\mathfrak{U}) \cap K_A$. Then $K_A(\mathfrak{U})$ is a closed subcomplex of $K_X(\mathfrak{U})$ and π_1^2 maps $K_A(\mathfrak{U}_1)$ into $K_A(\mathfrak{U}_2)$.

The p^{th} cohomology group of $K_X(\mathfrak{U})$ modulo $K_A(\mathfrak{U})$ based on infinite cochains will be denoted by $H^p(X, A; \mathfrak{U})$. Then π_1^2 induces a homomorphism

$$\pi_1^{2*} : H^p(X, A; \mathfrak{U}_1) \rightarrow H^p(X, A; \mathfrak{U}_2).$$

and the groups $H^p(X, A; \mathfrak{U})$ form a direct system under the mappings π^* . Define

$$\tilde{H}^p(X, A) = \varinjlim H^p(X, A; \mathfrak{U}).$$

Let $\phi \in \Phi_z^p(X, A)$. Then there is an open covering $\mathfrak{U}$ such that $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{U})$ and $\phi = 0$ on $N_{p+1}(\mathfrak{U}) \cap A^{p+1}$. Let $\phi(\mathfrak{U})$ denote ϕ restricted to the points of $N_{p+1}(\mathfrak{U})$. If the points of $N_{p+1}(\mathfrak{U})$ are identified with the p -simplices contained in $\mathfrak{U}$, $\phi(\mathfrak{U})$ may be considered as a cocycle of $K_X(\mathfrak{U})$ modulo $K_A(\mathfrak{U})$. Therefore, $\psi\phi(\mathfrak{U}) \in H^p(X, A; \mathfrak{U})$. Let $\{\psi\phi(\mathfrak{U})\}$ be the element of $\tilde{H}^p(X, A)$ determined by $\psi\phi(\mathfrak{U})$. If $\mathfrak{V}$ is another covering of X such that $\phi = 0$ on $N_{p+1}(\mathfrak{V}) \cap A^{p+1}$ and $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{V})$, let $\mathfrak{W}$ be a common refinement of both $\mathfrak{U}$ and $\mathfrak{V}$. Then $\phi = 0$ on $N_{p+1}(\mathfrak{W}) \cap A^{p+1}$ and $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{W})$. Let $\pi_1 : K_X(\mathfrak{W}) \rightarrow K_X(\mathfrak{U})$ and $\pi_2 : K_X(\mathfrak{W}) \rightarrow K_X(\mathfrak{V})$ be projections. Then it is clear that

$$\pi_1^* \psi\phi(\mathfrak{U}) = \psi\phi(\mathfrak{W}) = \pi_2^* \psi\phi(\mathfrak{V})$$

Therefore, $\{\psi\phi(\mathfrak{U})\} = \{\psi\phi(\mathfrak{V})\}$ so that $\{\psi\phi(\mathfrak{U})\}$ is independent of the choice of the covering $\mathfrak{U}$. If $\phi \in \Phi_z^p(X, A)$, define $\tau\phi = \{\psi\phi(\mathfrak{U})\}$. Then τ is a homomorphism $\tau : \Phi_z^p(X, A) \rightarrow \tilde{H}^p(X, A)$.

LEMMA 20.1. τ is onto.

PROOF: It is sufficient to show that for any open covering $\mathfrak{U}$ and cocycle c of

$K_X(\mathfrak{U}) \bmod K_A(\mathfrak{U})$ there is $\phi \in \Phi_z^p(X, A)$ such that $\phi\mathfrak{U} = c$. Considering c as a function defined on $N_{p+1}(\mathfrak{U})$, define ϕ to be c on $N_{p+1}(\mathfrak{U})$ and zero outside of $N_{p+1}(\mathfrak{U})$. Then $\phi = 0$ on A^{p+1} and $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{U})$ so that $\phi \in \Phi_z^p(X, A)$. It is clear that $\phi(\mathfrak{U}) = c$ for this ϕ .

LEMMA 20.2. *The kernel of $\tau = \Phi_B^p(X, A)$.*

PROOF: Let $\phi \in \Phi_z^p(X, A)$ such that $\tau\phi = 0$. Then there is an open covering $\mathfrak{B}$ such that $\bar{\phi} = 0$ on $N_{p+1}(\mathfrak{B}) \cap A^{p+1}$ and $\bar{\delta}\phi = 0$ on $N_{p+2}(\mathfrak{B})$ and $\phi(\mathfrak{B}) = \delta c$ where c is a $(p-1)$ -cochain of $K_X(\mathfrak{B}) \bmod K_A(\mathfrak{B})$. Define $\phi' \in \Phi^{p-1}(X, A)$ by

$$\phi'(x_0, \dots, x_{p-1}) = \begin{cases} c(x_0, \dots, x_{p-1}) & \text{if } (x_0, \dots, x_{p-1}) \in N_p(\mathfrak{B}) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{\delta}\phi' = \phi$ on $N_{p+1}(\mathfrak{B})$ so that $\phi \in \Phi_B^p(X, A)$. Therefore, the kernel of $\tau \subset \Phi_B^p(X, A)$. Since it is trivial that the kernel of $\tau \supset \Phi_B^p(X, A)$, the lemma is proved.

As a consequence of Lemmas 20.1 and 20.2 τ induces an isomorphism

$$\tau^*: H^p(X, A) \approx \tilde{H}^p(X, A)$$

satisfying $\tau^*(\gamma\phi) = \{\psi\phi(\mathfrak{U})\}$ for $\phi \in \Phi_z^p(X, A)$. Therefore $H^p(X, A)$ could have been defined to be the group $\tilde{H}^p(X, A)$. Since the definition of $\tilde{H}^p(X, A)$ is more complicated than the definition of $H^p(X, A)$, it seems preferable to use the construction of Section I for the cohomology groups.

The definition of $\tilde{H}^p(X, A)$ suggests a way of constructing homology groups for a compact coefficient group G' . Let the p^{th} homology group of $K_X(\mathfrak{U}) \bmod K_A(\mathfrak{U})$ with coefficient group G' and based on finite chains be denoted by $H_p(X, A, G'; \mathfrak{U})$. Then π_1^2 induces a homomorphism

$$\pi_{1*}^2: H_p(X, A, G'; \mathfrak{U}_1) \rightarrow H_p(X, A, G'; \mathfrak{U}_2)$$

for $\mathfrak{U}_1 > \mathfrak{U}_2$. The groups $H_p(X, A, G'; \mathfrak{U})$ form an inverse system under the mappings π_* . Define $\tilde{H}_p(X, A, G') = \varprojlim H_p(X, A, G'; \mathfrak{U})$.

It is possible to construct the homology groups directly from the functions. For the sake of simplicity, we consider only the absolute case (where $A = \emptyset$). Let X be a space and G' be a compact abelian group. If $\mathfrak{U}_\alpha$ is any open covering of X , f_p^α will denote a finitely non-zero function from $N_{p+1}(\mathfrak{U}_\alpha)$ to G' (that is, f_p^α is zero except for a finite set of points in $N_{p+1}(\mathfrak{U}_\alpha)$). ∂f_p^α will be a finitely non-zero function from $N_p(\mathfrak{U}_\alpha)$ to G' defined by

$$\partial f_0^\alpha = 0$$

$$\partial f_p^\alpha(x_0, \dots, x_{p-1}) = \sum (-1)^j f_p^\alpha(x_0, \dots, x_{j-1}, x, x_j, \dots, x_{p-1}) \text{ for } p > 0.$$

The summation is extended over all $x \in X$ and $0 \leq j \leq p-1$ such that $(x_0, \dots, x_{j-1}, x, x_j, \dots, x_{p-1}) \in N_{p+1}(\mathfrak{U}_\alpha)$. A p -cycle is a collection $\{f_p^\alpha\}$ with one function f_p^α for each covering $\mathfrak{U}_\alpha$ such that $\partial f_p^\alpha = 0$ on $N_p(\mathfrak{U}_\alpha)$ for each α and if $\mathfrak{U}_\alpha > \mathfrak{U}_\beta$ there is a finitely non-zero function $g_{p+1}^{\alpha\beta}$ from $N_{p+2}(\mathfrak{U}_\beta)$ to G' such that $f_p^\alpha - f_p^\beta = \partial g_{p+1}^{\alpha\beta}$ on $N_{p+1}(\mathfrak{U}_\beta)$. The p -cycles form an abelian group $Z_p(X; G')$ if

we define $\{f_p^\alpha\} - \{g_p^\alpha\} = \{f_p^\alpha - g_p^\alpha\}$. A p -cycle $\{f_p^\alpha\}$ is a p -boundary if for each α there is g_{p+1}^α such that $\partial g_{p+1}^\alpha = f_p^\alpha$. The boundaries form a subgroup $B_p(X; G')$ of $Z_p(X; G')$, and the p^{th} homology group is defined by

$$H_p(X; G') = Z_p(X; G')/B_p(X; G').$$

Then $H_p(X; G')$ is isomorphic to $\tilde{H}_p(X; G')$. This construction of $H_p(X; G')$ is quite similar to the Vietoris construction of the homology groups of a compact metric space [6].

APPENDIX B

21. Cup products

Let G_1, G_2 be two discrete abelian groups paired to a third such group G_3 . If $g_1 \in G_1, g_2 \in G_2$, let $g_1 \cdot g_2 \in G_3$ denote the pairing of g_1 and g_2 . If $\phi^p \in \Phi^p(X, A_1; G_1)$ and $\phi^q \in \Phi^q(X, A_2; G_2)$, then define the *cup product* $\phi^p \smile \phi^q \in \Phi^{p+q}(X, A_1 \cup A_2; G_3)$ by $(\phi^p \smile \phi^q)(x_0, \dots, x_{p+q}) = \phi^p(x_0, \dots, x_p) \phi^q(x_p, \dots, x_{p+q})$.

The following properties of the cup product are easily verified.

$$(21.1) \quad \phi^p \smile \phi^q \text{ pairs } \Phi^p(X, A_1; G_1) \text{ and } \Phi^q(X, A_2; G_2) \text{ to } \Phi^{p+q}(X, A_1 \cup A_2; G_3).$$

$$(21.2) \quad \text{If } G_1, G_2, G_3, G_{12}, G_{23}, G_{123} \text{ are six groups with pairings}$$

$$G_1, G_2 \text{ to } G_{12}$$

$$G_2, G_3 \text{ to } G_{23}$$

$$G_1, G_{23} \text{ to } G_{123}$$

$$G_{12}, G_3 \text{ to } G_{123}$$

such that $(G_1, G_2), G_3$ to G_{123} is the same as $G_1, (G_2, G_3)$ to G_{123} , then $(\phi^p \smile \phi^q) \smile \phi^r = \phi^p \smile (\phi^q \smile \phi^r)$ for $\phi^p \in \Phi^p(X, A_1; G_1), \phi^q \in \Phi^q(X, A_2; G_2), \phi^r \in \Phi^r(X, A_3; G_3)$.

$$(21.3) \quad \text{Let } \Gamma \text{ be the group of integers and consider it paired with any } G \text{ to } G \text{ in the obvious manner. Let } \phi^0 \in \Phi^0(X; \Gamma) \text{ be defined by } \phi^0(x) = 1 \text{ for all } x \in X. \text{ Then } \phi^0 \smile \phi^p = \phi^p \text{ and } \phi^p \smile \phi^0 = \phi^p \text{ for all } \phi^p \in \Phi^p(X, A; G).$$

$$(21.4) \quad \phi^p \smile \phi^q \text{ pairs } \Phi_0^p(X, A_1; G_1) \text{ and } \Phi_0^q(X, A_2; G_2) \text{ to } \Phi_0^{p+q}(X, A_1 \cup A_2; G_3).$$

$$(21.5) \quad \bar{\delta}(\phi^p \smile \phi^q) = \bar{\delta}\phi^p \smile \phi^q + (-1)^p \phi^p \smile \bar{\delta}\phi^q.$$

$$(21.6) \quad \text{The cup product can be extended to a pairing } h^p \smile h^q \text{ of } H^p(X, A_1; G_1) \text{ and } H^q(X, A_2; G_2) \text{ to } H^{p+q}(X, A_1 \cup A_2; G_3) \text{ such that } (\gamma\phi^p) \smile (\gamma\phi^q) = \gamma(\phi^p \smile \phi^q) \text{ for } \phi^p \in \Phi_z^p(X, A_1; G_1) \text{ and } \phi^q \in \Phi_z^q(X, A_2; G_2).$$

$$(21.7) \quad \text{If } f: X \rightarrow Y \text{ is continuous and } f(A_1) \subset B_1 \text{ and } f(A_2) \subset B_2, \text{ then } f^*(h^p \smile h^q) = (f^*h^p) \smile (f^*h^q) \text{ for } h^p \in H^p(Y, B_1; G_1) \text{ and } h^q \in H^q(Y, B_2; G_2).$$

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